

# PROPER ACTIONS ON CORANK-ONE REDUCTIVE HOMOGENEOUS SPACES

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**ABSTRACT.** Let  $\mathbf{k}$  be a local field,  $G$  the set of  $\mathbf{k}$ -points of a connected semisimple algebraic  $\mathbf{k}$ -group  $\mathbf{G}$ , and  $H$  the set of  $\mathbf{k}$ -points of a connected reductive algebraic  $\mathbf{k}$ -subgroup  $\mathbf{H}$  of  $\mathbf{G}$  such that  $\text{rank}_{\mathbf{k}}(\mathbf{H}) = \text{rank}_{\mathbf{k}}(\mathbf{G}) - 1$ . We consider discrete subgroups  $\Gamma$  of  $G$  acting properly discontinuously on  $G/H$  and we examine their images under a Cartan projection  $\mu : G \rightarrow V^+$ , where  $V^+$  is a closed convex cone in a real finite-dimensional vector space. We show that if  $\Gamma$  is neither a torsion group nor a virtually cyclic group, then  $\mu(\Gamma)$  is almost entirely contained in one connected component of  $V^+ \setminus C_H$ , where  $C_H$  denotes the convex hull of  $\mu(H)$  in  $V^+$ . As an application, we describe all torsion-free discrete subgroups of  $G \times G$  acting properly discontinuously on  $G$  by left and right translation when  $\text{rank}_{\mathbf{k}}(\mathbf{G}) = 1$ .

## 1. INTRODUCTION

Let  $\mathbf{k}$  be a local field,  $G$  the set of  $\mathbf{k}$ -points of a connected semisimple algebraic  $\mathbf{k}$ -group of rank one, and  $\Delta_G$  the diagonal of  $G \times G$ . In this paper we describe all torsion-free discrete subgroups of  $G \times G$  acting properly discontinuously on  $(G \times G)/\Delta_G$  (Theorem 1.3). To this end, we prove a general result on the Cartan projection of discrete groups acting properly discontinuously on corank-one reductive homogeneous spaces (Theorem 1.2). This result holds for algebraic groups over any local field, but we first state it in the setting of real Lie groups (Theorem 1.1).

**1.1. The main result in the real case.** Let  $G$  be a real connected semisimple linear Lie group and  $H$  a closed connected reductive subgroup of  $G$ . It is known that  $G$  contains an infinite discrete subgroup  $\Gamma$  acting properly discontinuously on  $G/H$  if and only if  $\text{rank}_{\mathbb{R}}(H) < \text{rank}_{\mathbb{R}}(G)$ ; this is the Calabi-Markus phenomenon ([Ko1], Cor. 4.4). In this paper we consider the case when  $\text{rank}_{\mathbb{R}}(H) = \text{rank}_{\mathbb{R}}(G) - 1$ .

Let us introduce some notation. Fix a Cartan subgroup  $A$  of  $G$  with Lie algebra  $\mathfrak{a}$ . Denote by  $\Phi = \Phi(A, G)$  the system of restricted roots of  $A$  in  $G$ , by  $\Phi^+$  a system of positive roots, by  $A^+ = \{a \in A, \chi(a) \geq 1 \ \forall \chi \in \Phi^+\}$  the corresponding closed positive Weyl chamber, and set  $V^+ = \log A^+ \subset \mathfrak{a}$ . There is a maximal compact subgroup  $K$  of  $G$  such that the Cartan decomposition  $G = KA^+K$  holds: every element  $g \in G$  may be written as  $g = k_1 a k_2$  for some  $k_1, k_2 \in K$  and a unique  $a \in A^+$  ([Hel], Chap. 9, Th. 1.1). Setting  $\mu(g) = \log a$  defines a map  $\mu : G \rightarrow V^+$ , which is continuous, proper, and surjective. It is called the *Cartan projection* relative to the Cartan decomposition  $G = KA^+K$ .

Since  $\text{rank}_{\mathbb{R}}(H) = \text{rank}_{\mathbb{R}}(G) - 1$ , the set  $\mu(H)$  separates  $V^+$  into finitely many connected components, which are permuted by the opposition involution  $\iota$ . (Recall that for every  $a \in A^+$  we have  $\iota(\log a) = \log a'$ , where  $a'$  is the unique element of  $A^+$  conjugate to  $a^{-1}$ .)

In this setting our main result is the following.

**Theorem 1.1.** *Let  $G$  be a real connected semisimple linear Lie group and  $H$  a closed connected reductive subgroup of  $G$  such that  $\text{rank}_{\mathbb{R}}(H) = \text{rank}_{\mathbb{R}}(G) - 1$ . For every discrete subgroup  $\Gamma$  of  $G$  acting properly discontinuously on  $G/H$ , there exists a connected component  $C$  of  $V^+ \setminus \mu(H)$  such that  $\mu(\gamma) \in C \cup \iota(C)$  for almost all  $\gamma \in \Gamma$ . If  $\Gamma$  is not virtually cyclic, then  $\iota(C) = C$ .*

Recall that a group  $\Gamma$  is said to satisfy some property *virtually* if it contains a subgroup of finite index satisfying this property. A property is said to be true for *almost all*  $\gamma \in \Gamma$  if it is true for all  $\gamma \in \Gamma$  with at most finitely many exceptions.

By results of Chevalley ([Che], Chap. 2, Th. 14 & 15), if  $G$  is a real connected semisimple linear Lie group and  $H$  a closed connected reductive subgroup of  $G$ , then  $G$  (resp.  $H$ ) is the identity component (for the real topology) of the set of  $\mathbb{R}$ -points of a connected semisimple linear algebraic  $\mathbb{R}$ -group  $\mathbf{G}$  (resp. of a connected reductive algebraic  $\mathbb{R}$ -subgroup  $\mathbf{H}$  of  $\mathbf{G}$ ). Theorem 1.1 is equivalent to the analogous result where  $G$  (resp.  $H$ ) is replaced by  $\mathbf{G}(\mathbb{R})$  (resp. by  $\mathbf{H}(\mathbb{R})$ ). We prove this result not only for  $\mathbb{R}$ -groups, but more generally for algebraic groups over any local field  $\mathbf{k}$ .

**1.2. The main result in the general case.** Let  $\mathbf{k}$  be a local field, *i.e.*,  $\mathbb{R}$ ,  $\mathbb{C}$ , a finite extension of  $\mathbb{Q}_p$ , or the field  $\mathbb{F}_q((t))$  of formal Laurent series over a finite field  $\mathbb{F}_q$ . Let  $G$  be the set of  $\mathbf{k}$ -points of a connected semisimple algebraic  $\mathbf{k}$ -group  $\mathbf{G}$  and  $H$  the set of  $\mathbf{k}$ -points of a connected reductive algebraic  $\mathbf{k}$ -subgroup  $\mathbf{H}$  of  $\mathbf{G}$  such that  $\text{rank}_{\mathbf{k}}(\mathbf{H}) = \text{rank}_{\mathbf{k}}(\mathbf{G}) - 1$ . There is a Cartan projection  $\mu$  of  $G$  to a closed convex cone  $V^+$  in some real finite-dimensional vector space (see Section 2). The convex hull  $C_H$  of  $\mu(H)$  in  $V^+$  separates  $V^+$  into finitely many connected components. The opposition involution  $\mu(G) \rightarrow \mu(G)$ , which maps  $\mu(g)$  to  $\mu(g^{-1})$  for all  $g \in G$ , extends to an involution  $\iota$  of  $V^+$  preserving  $C_H$  and permuting the connected components of  $V^+ \setminus C_H$  (see Section 3.1). Our main result in this general setting is the following.

**Theorem 1.2.** *Let  $\mathbf{k}$  be a local field,  $G$  the set of  $\mathbf{k}$ -points of a connected semisimple algebraic  $\mathbf{k}$ -group  $\mathbf{G}$ , and  $H$  the set of  $\mathbf{k}$ -points of a connected reductive algebraic  $\mathbf{k}$ -subgroup  $\mathbf{H}$  of  $\mathbf{G}$  such that  $\text{rank}_{\mathbf{k}}(\mathbf{H}) = \text{rank}_{\mathbf{k}}(\mathbf{G}) - 1$ . For every discrete subgroup  $\Gamma$  of  $G$  that acts properly discontinuously on  $G/H$  and that is not a torsion group, there exists a connected component  $C$  of  $V^+ \setminus C_H$  such that  $\mu(\gamma) \in C \cup \iota(C)$  for almost all  $\gamma \in \Gamma$ . If  $\Gamma$  is not virtually cyclic, then  $\iota(C) = C$ .*

When  $\mathbf{k}$  has characteristic zero, Theorem 1.2 holds without assuming that  $\Gamma$  is not a torsion group: indeed, in this case every discrete torsion subgroup of  $G$  is finite (Lemma 3.1). This is not true when  $\mathbf{k} = \mathbb{F}_q((t))$  for some finite field  $\mathbb{F}_q$ : in positive characteristic there are infinite discrete torsion subgroups of  $G$  that do not satisfy the conclusions of Theorem 1.2. We will give an example of such a group in Section 5.2.

**1.3. An application to  $(G \times G)/\Delta_G$ .** Our first application of Theorem 1.2, which is actually the main motivation of this paper, concerns homogeneous spaces of the form  $(G \times G)/\Delta_G$ , where  $G$  is the set of  $\mathbf{k}$ -points of a connected semisimple algebraic  $\mathbf{k}$ -group  $\mathbf{G}$  with  $\text{rank}_{\mathbf{k}}(\mathbf{G}) = 1$ , and where  $\Delta_G$  is the diagonal of  $G \times G$ . In this situation, if  $\mu$  is a Cartan projection of  $G$ , then  $\mu \times \mu$  is a Cartan projection of  $G \times G$ ; we identify  $V^+$  with  $\mathbb{R}^+ \times \mathbb{R}^+$  and  $C_H$  with the diagonal of  $\mathbb{R}^+ \times \mathbb{R}^+$ .

**Theorem 1.3.** *Let  $\mathbf{k}$  be a local field,  $G$  the set of  $\mathbf{k}$ -points of a connected semisimple algebraic  $\mathbf{k}$ -group  $\mathbf{G}$  with  $\text{rank}_{\mathbf{k}}(\mathbf{G}) = 1$ , and  $\Delta_G$  the diagonal of  $G \times G$ . Let  $\Gamma$  be a discrete subgroup of  $G \times G$ .*

- (1) *Assume that  $\Gamma$  is torsion-free. Then it acts properly discontinuously on  $(G \times G)/\Delta_G$  if and only if, up to switching the factors of  $G \times G$ , it is a graph of the form*

$$\{(\gamma, \varphi(\gamma)), \gamma \in \Gamma_0\},$$

*where  $\Gamma_0$  is a discrete subgroup of  $G$  and  $\varphi : \Gamma_0 \rightarrow G$  is a group homomorphism such that for all  $R > 0$ , almost all  $\gamma \in \Gamma_0$  satisfy  $\mu(\varphi(\gamma)) < \mu(\gamma) - R$ .*

- (2) *Assume that  $\Gamma$  is residually finite and is not a torsion group. Then it acts properly discontinuously on  $(G \times G)/\Delta_G$  if and only if, up to switching the factors of  $G \times G$ , it has a finite-index subgroup  $\Gamma'$  that is a graph as in (1).*

Note that  $(g, h)\Delta_G \mapsto gh^{-1}$  defines a  $(G \times G)$ -equivariant isomorphism from  $(G \times G)/\Delta_G$  to  $G$ , where  $G \times G$  acts on  $G$  by  $(g_1, g_2) \cdot g = g_1 g_2^{-1}$ . Thus Theorem 1.3 describes all torsion-free discrete subgroups of  $G \times G$  acting properly discontinuously on  $G$  by left and right translation.

Recall that a group is said to be *residually finite* if the intersection of its normal finite-index subgroups is trivial. It is known that if  $\Gamma \subset G \times G$  is finitely generated, then it is residually finite (see [Alp], Cor. 1); if moreover  $\mathbf{k}$  has characteristic zero, then  $\Gamma$  has a finite-index subgroup that is torsion-free by Selberg's lemma ([Sel], Lem. 8).

In the case of  $G = \text{PSL}_2(\mathbb{R})$ , Theorem 1.3 has been proved for torsion-free groups by Kulkarni and Raymond [KR]. In [Ko2], Kobayashi considered the more general case when  $G$  is a real connected semisimple linear Lie group with  $\text{rank}_{\mathbb{R}}(G) = 1$ : he showed that every torsion-free discrete subgroup of  $G \times G$  acting properly discontinuously on  $(G \times G)/\Delta_G$  is a graph, and asked whether one of the two projections of this graph is always discrete in  $G$ . Theorem 1.3 above answers this question positively and generalizes Kobayashi's result to all local fields. It gives a complete description of all torsion-free discrete subgroups of  $G \times G$  acting properly discontinuously on  $(G \times G)/\Delta_G$  in terms of a Cartan projection of  $G$ .

Theorem 1.3 applies to three-dimensional compact *anti-de Sitter* manifolds, *i.e.*, to three-dimensional compact Lorentz manifolds with constant sectional curvature  $-1$ . Indeed, such manifolds are modeled on

$$\text{AdS}^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4, \quad x_1^2 + x_2^2 - x_3^2 - x_4^2 = 1\}$$

endowed with the Lorentz metric induced by  $x_1^2 + x_2^2 - x_3^2 - x_4^2$ , which identifies with  $(\text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R}))/\Delta_{\text{SL}_2(\mathbb{R})}$  (see Section 5.3). Since three-dimensional

compact anti-de Sitter manifolds are complete [Kli], they are quotients of the universal covering of  $\text{AdS}^3$ . By [KR], up to a finite covering, they may in fact be written as

$$\Gamma \backslash (\text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R})) / \Delta_{\text{PSL}_2(\mathbb{R})},$$

where  $\Gamma$  is a torsion-free discrete subgroup of  $\text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R})$  acting properly discontinuously on  $(\text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R})) / \Delta_{\text{PSL}_2(\mathbb{R})}$ . We refer the reader to the introduction of [Sal] for more details.

More generally, for any local field  $\mathbf{k}$  and any quadratic form  $Q$  of Witt index two on  $\mathbf{k}^4$ , the quadric

$$S(Q) = \{x \in \mathbf{k}^4, Q(x) = 1\}$$

identifies with  $(\text{SL}_2(\mathbf{k}) \times \text{SL}_2(\mathbf{k})) / \Delta_{\text{SL}_2(\mathbf{k})}$  (see Section 5.3). Theorem 1.3 therefore applies to the discrete subgroups of  $\text{SL}_2(\mathbf{k}) \times \text{SL}_2(\mathbf{k})$  acting properly discontinuously on  $S(Q)$ .

Note that Theorem 1.3 cannot be generalized to groups  $G$  of higher rank. Indeed, take for instance  $G = \text{SO}(2, 2n)$ , and let  $\Gamma_1$  (resp.  $\Gamma_2$ ) be a torsion-free discrete subgroup of  $\text{SO}(1, 2n)$  (resp. of  $\text{U}(1, n)$ ), where  $\text{SO}(1, 2n)$  (resp.  $\text{U}(1, n)$ ) is seen as a subgroup of  $G$ . By [Ko1], Prop. 4.9,  $\Gamma_1 \times \Gamma_2$  acts properly discontinuously on  $(G \times G) / \Delta_G$ . Other examples are obtained by replacing the triple  $(\text{SO}(2, 2n), \text{SO}(1, 2n), \text{U}(1, n))$  by  $(\text{SO}(4, 4n), \text{SO}(3, 4n), \text{Sp}(1, n))$  or by  $(\text{U}(2, 2n), \text{U}(1) \times \text{U}(1, n), \text{Sp}(1, n))$  (see [Ko1]).

**1.4. An application to  $\text{SL}_n(\mathbf{k}) / \text{SL}_{n-1}(\mathbf{k})$ .** As another application of Theorem 1.2, we give a simpler proof of the following result, due to Benoist [Be1].

**Corollary 1.4.** *Let  $\mathbf{k}$  be a local field of characteristic zero. If  $n \geq 3$  is odd, then every discrete subgroup of  $\text{SL}_n(\mathbf{k})$  acting properly discontinuously on  $\text{SL}_n(\mathbf{k}) / \text{SL}_{n-1}(\mathbf{k})$  is virtually abelian.*

Theorem 1.2 actually implies a slightly stronger version of Corollary 1.4: we may replace “virtually abelian” by “virtually cyclic”.

One consequence of Corollary 1.4 is that in characteristic zero if  $n \geq 3$  is odd, then the homogeneous space  $\text{SL}_n(\mathbf{k}) / \text{SL}_{n-1}(\mathbf{k})$  has no compact quotient, *i.e.*, there is no discrete subgroup  $\Gamma$  of  $\text{SL}_n(\mathbf{k})$  acting properly discontinuously on  $\text{SL}_n(\mathbf{k}) / \text{SL}_{n-1}(\mathbf{k})$  with  $\Gamma \backslash \text{SL}_n(\mathbf{k}) / \text{SL}_{n-1}(\mathbf{k})$  compact (see [Be1]).

**1.5. Organization of the paper.** In Section 2 we recall basic facts about Bruhat-Tits buildings, Cartan decompositions, and Cartan projections. Section 3 is devoted to the proof of Theorem 1.2; we also discuss the assumption that  $\Gamma$  is not a torsion group. In Section 4 we show how Theorem 1.2 implies Corollary 1.4 in the case of  $G = \text{SL}_n(\mathbf{k})$  and  $H = \text{SL}_{n-1}(\mathbf{k})$ . In Section 5 we prove Theorem 1.3; we also show that the hypothesis that  $\Gamma$  is not a torsion group is necessary in positive characteristic, and we describe our application to three-dimensional quadrics.

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## 2. CARTAN PROJECTIONS

Throughout this article, we denote by  $\mathbf{k}$  a local field, *i.e.*,  $\mathbb{R}$ ,  $\mathbb{C}$ , a finite extension of  $\mathbb{Q}_p$ , or the field  $\mathbb{F}_q((t))$  of formal Laurent series over a finite field  $\mathbb{F}_q$ . If  $\mathbf{k} = \mathbb{R}$  or  $\mathbb{C}$ , we denote by  $|\cdot|$  the usual absolute value on  $\mathbf{k}$ ; we set  $\mathbf{k}^+ = [1, +\infty[$ . If  $\mathbf{k}$  is nonarchimedean, we denote by  $\mathcal{O}$  the ring of integers of  $\mathbf{k}$ , by  $q$  the cardinal of the residue field of  $\mathbf{k}$ , by  $\omega$  the (additive) valuation on  $\mathbf{k}$  sending any uniformizer to 1, and by  $|\cdot| = q^{-\omega(\cdot)}$  the corresponding (multiplicative) absolute value; we set  $\mathbf{k}^+ = \{x \in \mathbf{k}, |x| \geq 1\}$ . If  $\mathbf{G}$  is an algebraic group, we denote by  $G$  the set of its  $\mathbf{k}$ -points and by  $\mathfrak{g}$  the Lie algebra of  $G$ .

In this section, we recall a few well-known facts on connected semisimple algebraic  $\mathbf{k}$ -groups and their Cartan projections.

**2.1. Weyl chambers.** Fix a connected semisimple algebraic  $\mathbf{k}$ -group  $\mathbf{G}$ . Recall that the  $\mathbf{k}$ -split  $\mathbf{k}$ -tori of  $\mathbf{G}$  are all conjugate over  $\mathbf{k}$  ([BoT], Th. 4.21). Fix such a torus  $\mathbf{A}$  and let  $\mathbf{N}$  (resp.  $\mathbf{Z}$ ) denote its normalizer (resp. centralizer) in  $\mathbf{G}$ . The group  $X(\mathbf{A})$  of  $\mathbf{k}$ -characters of  $\mathbf{A}$  and the group  $Y(\mathbf{A})$  of  $\mathbf{k}$ -cocharacters are both free  $\mathbb{Z}$ -modules of rank  $r = \text{rank}_{\mathbf{k}}(\mathbf{G})$ , and there is a perfect pairing

$$\langle \cdot, \cdot \rangle : X(\mathbf{A}) \times Y(\mathbf{A}) \longrightarrow \mathbb{Z}.$$

If  $\mathbf{k}$  is nonarchimedean, we set  $A^\circ = A$ ; if  $\mathbf{k} = \mathbb{R}$  or  $\mathbb{C}$ , we set

$$A^\circ = \{a \in A, \quad \chi(a) \in ]0, +\infty[ \quad \forall \chi \in X(\mathbf{A})\}.$$

The set  $\Phi = \Phi(\mathbf{A}, \mathbf{G})$  of restricted roots of  $\mathbf{A}$  in  $\mathbf{G}$ , *i.e.*, the set of nontrivial weights of  $\mathbf{A}$  in the adjoint representation of  $\mathbf{G}$ , is a root system of the real vector space  $V = Y(\mathbf{A}) \otimes_{\mathbb{Z}} \mathbb{R}$  ([BoT], Cor. 5.8). The group  $W = N/Z$  is finite and identifies with the Weyl group of  $\Phi$  ([BoT], § 5.1 & Th. 5.3). Choose a basis  $\Delta = \{\alpha_1, \dots, \alpha_r\}$  of  $\Phi$  and let

$$\begin{aligned} A^+ &= \{a \in A^\circ, \quad \alpha_i(a) \in \mathbf{k}^+ \quad \forall 1 \leq i \leq r\} \\ (\text{resp. } V^+ &= \{x \in V, \quad \langle \alpha_i, x \rangle \geq 0 \quad \forall 1 \leq i \leq r\}) \end{aligned}$$

denote the closed positive Weyl chamber in  $A^\circ$  (resp. in  $V$ ) corresponding to  $\Delta$ ; the set  $V^+$  is a closed convex cone in  $V$ . If  $\mathbf{k} = \mathbb{R}$  or  $\mathbb{C}$ , then  $V$  identifies with  $\mathfrak{a}$  and  $V^+$  with  $\log A^+ \subset \mathfrak{a}$ ; we endow  $V$  with the Euclidean norm  $\|\cdot\|$  induced by the Killing form of  $\mathfrak{g}$ . If  $\mathbf{k}$  is nonarchimedean, we endow  $V$  with any  $W$ -invariant Euclidean norm  $\|\cdot\|$ .

**2.2. The Bruhat-Tits building.** In this subsection we assume  $\mathbf{k}$  to be nonarchimedean. We briefly recall the construction of the Bruhat-Tits building of  $G$ , which is a metric space on which  $G$  acts properly discontinuously by isometries with a compact fundamental domain. We refer to the original articles [BT1] and [BT2], but the reader may also find [Rou] useful.

Let  $\text{Res}$  denote the restriction homomorphism from  $X(\mathbf{Z})$  to  $X(\mathbf{A})$ , where  $X(\mathbf{Z})$  denotes the group of  $\mathbf{k}$ -characters of  $\mathbf{Z}$ . There is a unique group homomorphism  $\nu : Z \rightarrow V$  such that

$$\langle \text{Res}(\chi), \nu(z) \rangle = -\omega(\chi(z))$$

for all  $\chi \in X(\mathbf{Z})$  and  $z \in Z$ . The set  $\nu(Z)$  is a lattice in  $V$ , and  $\nu(A)$  is a sublattice of  $\nu(Z)$  of finite index. The action of  $Z$  on  $V$  by translation

along  $\nu(Z)$  extends to an action of  $N$  on  $V$  by affine isometries; such an extension is unique up to translation.

For every  $\alpha \in \Phi$ , let  $U_\alpha$  denote the connected unipotent  $\mathbf{k}$ -subgroup of  $\mathbf{G}$  corresponding to the root  $\alpha$ , as defined in [BT2]; the Lie algebra of  $U_\alpha$  is  $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ , where  $\mathfrak{g}_{i\alpha}$  is the subspace of elements  $X \in \mathfrak{g}$  such that  $\text{Ad}(a)(X) = \alpha(a)^i X$  for all  $a \in A$ . For every  $u \in U_\alpha$ ,  $u \neq 1$ , the set  $N \cap U_{-\alpha} u U_{-\alpha}$  has a unique element, which acts on  $V$  by the orthogonal reflection in some affine hyperplane  $\mathcal{H}_u$ , defined by an equation of the form  $\langle \alpha, x \rangle + \psi_\alpha(u) = 0$ , where  $\psi_\alpha(u) \in \mathbb{R}$ . For every  $x \in V$ , set

$$U_{\alpha,x} = \{u \in U_\alpha, \quad u = 1 \text{ or } \langle \alpha, x \rangle + \psi_\alpha(u) \geq 0\};$$

by [BT2] it is a subgroup of  $U_\alpha$ . Set  $N_x = \{n \in N, \quad n \cdot x = x\}$  and let  $K_x$  denote the subgroup of  $G$  generated by  $N_x$  and the subgroups  $U_{\alpha,x}$ , where  $\alpha \in \Phi$ . The group  $K_x$  is a maximal compact open subgroup of  $G$ .

With this notation, the *Bruhat-Tits building*  $X$  of  $G$  is the set of equivalence classes of  $G \times V$  for the relation

$$(g, x) \sim (g', x') \iff \exists n \in N \text{ such that } x' = n \cdot x \text{ et } g^{-1}g'n \in K_x.$$

We endow  $X$  with the quotient topology induced by the discrete topology of  $G$  and the Euclidean structure of  $V$ . By construction,  $V$  embeds into  $X$ ; we identify it with its image in  $X$ . The group  $G$  acts on  $X$  by

$$g' \cdot \overline{(g, x)} = \overline{(g'g, x)},$$

where  $\overline{(g, x)}$  denotes the image of  $(g, x) \in G \times V$  in  $X$ . This action is properly discontinuous, with a compact fundamental domain. By construction, the stabilizer of any point  $x \in V$  is  $K_x$ . The *apartments* of  $X$  are the sets  $g \cdot V$ , where  $g \in G$ ; the *walls* of  $X$  are the sets  $g \cdot \mathcal{H}_u$ , where  $g \in G$  and  $u \in U_\alpha$  for some  $\alpha \in \Phi$ . A *chamber* of  $X$  (or *alcove*) is a connected component of  $X$  deprived of its walls. The space  $X$  has the following property: for any pair  $(x, x')$  of points in  $X$ , there is an apartment containing both  $x$  and  $x'$ . We can therefore endow  $X$  with a distance  $d$  defined as follows:  $d(x, x')$  is the Euclidean distance between  $x$  and  $x'$  in any apartment containing  $x$  and  $x'$  (it does not depend on the apartment). The group  $G$  acts on  $X$  by isometries for this distance.

**2.3. Cartan decompositions and Cartan projections.** If  $\mathbf{k} = \mathbb{R}$  or  $\mathbb{C}$ , then there is a maximal compact subgroup  $K$  of  $G$  such that the Cartan decomposition  $G = KA^+K$  holds: for every  $g \in G$ , there are elements  $k_1, k_2 \in K$  and a unique  $a \in A^+$  such that  $g = k_1 a k_2$  ([Hel], Chap. 9, Th. 1.1). Setting  $\mu(g) = \log a$  defines a map  $\mu : G \rightarrow V^+ \simeq \log A^+$ , which is continuous, proper, and surjective. It is called the *Cartan projection* relative to the Cartan decomposition  $G = KA^+K$ .

Now assume  $\mathbf{k}$  to be nonarchimedean. Consider the extremal point  $x_0$  of the closed cone  $V^+$ , defined by  $\langle \alpha_i, x_0 \rangle = 0$  for all  $1 \leq i \leq r$ , and set  $K = K_{x_0}$ . Let  $Z^+ \subset Z$  denote the inverse image of  $V^+$  under  $\nu$ . By [BT1] the group  $G$  acts transitively on the set of couples  $(\mathcal{A}, \mathcal{C})$ , where  $\mathcal{A}$  is an apartment of  $X$  and  $\mathcal{C}$  is a chamber of  $X$  contained in  $\mathcal{A}$ . This can be translated into algebraic terms by the existence of a *Cartan decomposition*  $G = KZ^+K$ : for every  $g \in G$  there are elements  $k_1, k_2 \in K$  and  $z \in Z^+$  such that  $g = k_1 z k_2$ , and  $\nu(z)$  is uniquely defined. Setting  $\mu(g) = \nu(z)$

defines a map  $\mu : G \rightarrow V^+$ , which is continuous and proper; its image  $\mu(G)$  is the intersection of  $V^+$  with a lattice of  $V$ . The map  $\mu$  is called the *Cartan projection* relative to the Cartan decomposition  $G = KZ^+K$ .

**2.4. A geometric interpretation.** Let  $X$  be either the Riemannian symmetric space  $G/K$  if  $\mathbf{k} = \mathbb{R}$  or  $\mathbb{C}$ , or the Bruhat-Tits building of  $G$  if  $\mathbf{k}$  is nonarchimedean. We now recall a geometric interpretation of the Cartan projection  $\mu$  in terms of a distance on  $X$ .

Assume that  $\mathbf{k} = \mathbb{R}$  or  $\mathbb{C}$ , and let  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$  be the Cartan decomposition of  $\mathfrak{g}$  corresponding to the Cartan decomposition  $G = KA^+K$ . The Killing form  $\kappa$  of  $\mathfrak{g}$  is definite positive on  $\mathfrak{p}$ , hence induces a Euclidean norm  $\|\cdot\|$  on  $\mathfrak{p}$ . Let  $\pi$  denote the natural projection of  $G$  onto  $X = G/K$ , and set  $x_0 = \pi(1) \in X$ . The map  $d\pi_1$  realizes an isomorphism between  $\mathfrak{p}$  and the tangent space of  $X$  at  $x_0$ ; thus  $\kappa|_{\mathfrak{p} \times \mathfrak{p}}$  induces a  $G$ -invariant Riemannian metric on  $X$ . Let  $d$  denote the corresponding distance on  $X$ . The following result is probably well known; we prove it for the reader's convenience.

**Lemma 2.1** ( $\mathbf{k} = \mathbb{R}$  or  $\mathbb{C}$ ). *Let  $\rho : X \rightarrow V^+$  denote the map sending  $x = g \cdot x_0 \in X$  to  $\mu(g)$ . For all  $x, x' \in X$ ,*

$$\|\rho(x) - \rho(x')\| \leq d(x, x').$$

*Moreover, the restriction of  $\rho$  to  $A^+ \cdot x_0$  is an isometry.*

*Proof.* We identify  $V^+$  with  $\log A^+ \subset \mathfrak{a}$ . Let  $\text{Exp} : \mathfrak{p} \rightarrow X$  denote the exponential diffeomorphism mapping  $Y \in \mathfrak{p}$  to  $\gamma_Y(1)$ , where  $\gamma_Y$  is the unique geodesic in  $X$  such that  $\gamma_Y(0) = x_0$  and  $\gamma'_Y(0) = d\pi_1(Y)$ . For every  $x \in X$ , there exists  $k \in K$  such that  $x = k \exp(\rho(x)) \cdot x_0$ ; by [Hel], Chap. 4, Th. 3.3,

$$(2.1) \quad x = \text{Exp}((\text{Ad } k)(\rho(x))).$$

Fix  $x, x' \in X$  and let  $\gamma = (y_t)_{t \in [0,1]}$  be the geodesic segment from  $y_0 = x$  to  $y_1 = x'$ . By [Hel], p. 295, and (2.1), the map  $t \mapsto \rho(y_t)$  is smooth and there exists a smooth map  $t \mapsto k_t$  from  $[0,1]$  to  $K$  such that  $y_t = \text{Exp}((\text{Ad } k_t)(\rho(y_t)))$  for all  $t \in [0,1]$ . Since  $X$  has nonpositive sectional curvature ([Hel], Chap. 5, Th. 3.1), the length of  $\gamma$  in  $X$  is not less than the length of  $\text{Exp}^{-1}(\gamma)$  in  $\mathfrak{p}$  ([Hel], Chap. 1, Th. 13.1), namely,

$$(2.2) \quad d(x, x') \geq \int_0^1 \left\| \frac{d((\text{Ad } k_t)(\rho(y_t)))}{dt}(t') \right\| dt'.$$

Now for all  $t' \in [0,1]$ ,

$$\frac{d((\text{Ad } k_t)(\rho(y_t)))}{dt}(t') = (\text{Ad } k_{t'}) \left( \frac{d(\rho(y_t))}{dt}(t') \right) + \left( \frac{d(\text{Ad } k_t)}{dt}(t') \right) (\rho(y_{t'})),$$

where

$$(\text{Ad } k_{t'}) \left( \frac{d(\rho(y_t))}{dt}(t') \right) \in (\text{Ad } k_{t'}) (\mathfrak{a})$$

and

$$\begin{aligned} \left( \frac{d(\text{Ad } k_t)}{dt}(t') \right) (\rho(y_{t'})) &= (\text{Ad } k_{t'}) \left( \text{ad} \left( \frac{d(k_{t'}^{-1} k_{t'+t})}{dt}(0) \right) (\rho(y_{t'})) \right) \\ &\in (\text{Ad } k_{t'}) ([\mathfrak{k}, \mathfrak{a}]). \end{aligned}$$

The subspaces  $\mathfrak{a}$  and  $[\mathfrak{k}, \mathfrak{a}]$  are orthogonal with respect to  $\kappa$ . Indeed, the decomposition of  $\mathfrak{g}$  into eigenspaces under the adjoint action of  $\mathfrak{a}$  is orthogonal with respect to  $\kappa$  ([Hel], Chap. 3, Th. 4.2); in particular,  $\mathfrak{a}$  is orthogonal to the sum  $[\mathfrak{g}, \mathfrak{a}]$  of the root spaces of  $\mathfrak{g}$ . Since  $\kappa$  is invariant under  $\text{Ad } G$  ([Hel], p. 131), the subspaces  $(\text{Ad } k_{t'}) (\mathfrak{a})$  and  $(\text{Ad } k_{t'}) ([\mathfrak{k}, \mathfrak{a}])$  are orthogonal with respect to  $\kappa$  and

$$(2.3) \quad \left\| \frac{d((\text{Ad } k_t)(\rho(y_t)))}{dt}(t') \right\| \geq \left\| (\text{Ad } k_{t'}) \left( \frac{d(\rho(y_t))}{dt}(t') \right) \right\| \\ = \left\| \frac{d(\rho(y_t))}{dt}(t') \right\|.$$

Thus

$$d(x, x') \geq \int_0^1 \left\| \frac{d(\rho(y_t))}{dt}(t') \right\| dt' = \|\rho(x) - \rho(x')\|.$$

If  $x, x' \in A^+ \cdot x_0$ , then  $k_t = 1$  for all  $t \in [0, 1]$ ; hence (2.3) is an equality. Moreover, in this case (2.2) is also an equality since the geodesic submanifold  $A \cdot x_0 = \text{Exp}(\mathfrak{a})$  has zero sectional curvature ([Hel], Chap. 5, § 3, Rem. 2). This implies  $d(x, x') = \|\rho(x) - \rho(x')\|$ .  $\square$

Since  $K$  fixes  $x_0$  and since  $G$  acts on  $X$  by isometries, Lemma 2.1 implies that for every  $a \in A^+$  and every  $g \in KaK$ ,

$$(2.4) \quad d(g \cdot x_0, x_0) = d(a \cdot x_0, x_0) = \|\rho(a \cdot x_0) - \rho(x_0)\| = \|\mu(g)\|.$$

Now assume  $\mathbf{k}$  to be nonarchimedean and let  $X$  denote the Bruhat-Tits building of  $G$ , endowed with the distance  $d$  defined in Section 2.2. Recall that  $K = K_{x_0}$  is the stabilizer of the point  $x_0 \in V$  defined by  $\langle \alpha_i, x_0 \rangle = 0$  for all  $1 \leq i \leq r$ . Since  $G$  acts on  $X$  by isometries and since  $V$  is isometrically embedded as an apartment in  $X$ , for every  $z \in Z^+$  and every  $g \in KzK$ ,

$$(2.5) \quad d(g \cdot x_0, x_0) = d(z \cdot x_0, x_0) = d(\mu(g), x_0) = \|\mu(g)\|,$$

where  $\|\cdot\|$  is the Euclidean norm on  $V$ . Lemma 2.1 also holds in this setting.

**Lemma 2.2** ( $\mathbf{k}$  nonarchimedean). *Let  $\rho : X \rightarrow V^+$  denote the map sending  $x = g \cdot x_0 \in X$  to  $\mu(g)$ . For all  $x, x' \in X$ ,*

$$\|\rho(x) - \rho(x')\| \leq d(x, x').$$

*Proof.* Let  $\mathcal{C}$  denote the unique chamber in  $V^+$  containing  $x_0$ . We first recall the construction of a retraction  $\rho_{V, \mathcal{C}} : X \rightarrow V$ , as defined in [BT1], § 2.3. For every  $x \in X$ , there is an apartment  $\mathcal{A}$  containing both  $x$  and  $\mathcal{C}$  ([BT1], Prop. 2.3.1), and there is an element  $k \in K$  fixing  $\mathcal{C}$  pointwise and mapping  $\mathcal{A}$  to  $V$  ([BT1], Prop. 2.3.2). The point  $k \cdot x \in V$  does not depend on the choice of  $\mathcal{A}$  and  $k$ . Setting  $\rho_{V, \mathcal{C}}(x) = k \cdot x$  defines a map  $\rho_{V, \mathcal{C}} : X \rightarrow V$  such that for all  $x, x' \in X$ ,

$$\|\rho_{V, \mathcal{C}}(x) - \rho_{V, \mathcal{C}}(x')\| \leq d(x, x')$$

([BT1], Prop. 2.5.3). We claim that for all  $x, x' \in X$ ,

$$(2.6) \quad \|\rho(x) - \rho(x')\| \leq \|\rho_{V, \mathcal{C}}(x) - \rho_{V, \mathcal{C}}(x')\|.$$

Indeed, it follows from the definitions of  $\rho$  and  $\rho_{V, \mathcal{C}}$  that  $\rho_{V, \mathcal{C}}(x) \in W \cdot \rho(x)$  for all  $x \in X$ . Since the norm  $\|\cdot\|$  is  $W$ -invariant, it is enough to show that

$$(2.7) \quad \|\rho(x) - \rho(x')\| \leq \|\rho(x) - w \cdot \rho(x')\|$$



for all  $x, x' \in X$  and all  $w \in W$ . Recall that  $W$  is generated by the set  $S$  of orthogonal reflections in the hyperplanes  $\{x \in V, \langle \alpha_i, x \rangle = 0\}$ , where  $1 \leq i \leq r$ . Write  $w = s_m \dots s_1$ , where  $s_j \in S$  for all  $j$ . We argue by induction on  $m$ . If  $(s_m \dots s_1) \cdot \rho(x') \in V^+$ , then  $s_m \dots s_1 = 1$  and (2.7) is obvious. Otherwise, the points  $\rho(x)$  and  $(s_m \dots s_1) \cdot \rho(x')$  lie in two distinct connected components of  $V \setminus \mathcal{H}$ , where  $\mathcal{H}$  denotes the hyperplane of fixed points of  $s_m$ . Let  $y$  be the intersection point of  $\mathcal{H}$  with the line segment  $[\rho(x), (s_m \dots s_1) \cdot \rho(x')]$ . Since  $s_m$  is an orthogonal reflection,

$$\begin{aligned} \|\rho(x) - (s_m \dots s_1) \cdot \rho(x')\| &= \|\rho(x) - y\| + \|y - (s_m \dots s_1) \cdot \rho(x')\| \\ &= \|\rho(x) - y\| + \|y - (s_{m-1} \dots s_1) \cdot \rho(x')\| \\ &\geq \|\rho(x) - (s_{m-1} \dots s_1) \cdot \rho(x')\|. \end{aligned}$$

By the induction assumption,  $\|\rho(x) - (s_m \dots s_1) \cdot \rho(x')\| \geq \|\rho(x) - \rho(x')\|$ . This proves (2.6) and completes the proof of Lemma 2.2.  $\square$

The following result will be needed in the proof of Theorem 1.2.

**Lemma 2.3.** *Let  $\mathbf{k}$  be a local field,  $G$  the set of  $\mathbf{k}$ -points of a connected semisimple algebraic  $\mathbf{k}$ -group, and  $\mu : G \rightarrow V^+$  a Cartan projection. For all  $g, g' \in G$ , the following two inequalities hold:*

$$(2.8) \quad \|\mu(gg') - \mu(g)\| \leq \|\mu(g')\|,$$

$$(2.9) \quad \|\mu(gg') - \mu(g')\| \leq \|\mu(g)\|.$$

*Proof.* Since  $G$  acts on  $X$  by isometries, (2.8) follows immediately from Lemmas 2.1 and 2.2, together with Formulas (2.4) and (2.5). We claim that (2.8) implies (2.9). Indeed, if  $w \in W$  denotes the “longest” element of  $W$ , such that  $w \cdot z^{-1} \in Z^+$  for all  $z \in Z^+$ , then  $\mu(g^{-1}) = w \cdot (-\mu(g))$  for all  $g \in G$ . Since the norm  $\|\cdot\|$  on  $V$  is  $W$ -invariant, the opposition involution  $\iota : \mu(G) \rightarrow \mu(G)$ , which maps  $\mu(g)$  to  $\mu(g^{-1})$  for all  $g \in G$ , is an isometry. Together with (2.8), this implies

$$\|\mu(gg') - \mu(g')\| = \|\mu(g'^{-1}g^{-1}) - \mu(g'^{-1})\| \leq \|\mu(g^{-1})\| = \|\mu(g)\|. \quad \square$$

### 3. PROPER ACTIONS ON $G/H$ IN THE CORANK-ONE CASE

In this section we give a proof of Theorem 1.2 and we discuss the assumption that  $\Gamma$  is not a torsion group.

**3.1. Proof of Theorem 1.2.** With the notation of Section 2, let  $\mathbf{H}$  be a connected reductive algebraic  $\mathbf{k}$ -subgroup of  $\mathbf{G}$  with  $\text{rank}_{\mathbf{k}}(\mathbf{H}) = \text{rank}_{\mathbf{k}}(\mathbf{G}) - 1$ . Fix a maximal  $\mathbf{k}$ -split  $\mathbf{k}$ -torus  $\mathbf{A}_{\mathbf{H}}$  of  $\mathbf{H}$ . After conjugating  $\mathbf{H}$  by an element of  $G$ , we may assume that  $\mathbf{A}_{\mathbf{H}} \subset \mathbf{A}$  ([BoT], Th. 4.21). Recall that  $\mathbf{H}$  is the almost product of a central torus and of its derived group, which is semisimple ([BoT], Prop. 2.2). Therefore  $H$  admits a Cartan decomposition  $H = K_H Z_H^+ K_H$ , where  $Z_{\mathbf{H}}$  is the centralizer of  $\mathbf{A}_{\mathbf{H}}$  in  $\mathbf{H}$  and  $K_H$  is some maximal compact subgroup of  $H$ . We now use a result proved by Mostow [Mos] and Karpelevich [Kar] in the real case, and by Landvogt [Lan] in the nonarchimedean case: after conjugating  $\mathbf{H}$  by an element of  $G$ , we may assume that  $K_H \subset K$ . Thus  $\mu(H) = \mu(Z_H)$  and the convex hull  $C_H$  of  $\mu(H)$  in  $V^+$  is the intersection of  $V^+$  with a finite union of hyperplanes of  $V$  parametrized by the Weyl group  $W$ . The opposition involution

$\iota : \mu(G) \rightarrow \mu(G)$ , which maps  $\mu(g)$  to  $\mu(g^{-1})$  for all  $g \in G$ , extends to an isometry of  $V^+$ , still denoted by  $\iota$ . It preserves  $\mu(H)$ , hence  $C_H$ , and permutes the connected components of  $V^+ \setminus C_H$ .

Our proof of Theorem 1.2 is based on the *properness criterion* of Benoist ([Be1], Cor. 5.2) and Kobayashi ([Ko3], Th. 1.1), which states that a subgroup  $\Gamma$  of  $G$  acts properly discontinuously on  $G/H$  if and only if the set  $\mu(\Gamma) \cap (\mu(H) + C')$  is bounded for every compact subset  $C'$  of  $V$ . This condition is equivalent to the boundedness of  $\mu(\Gamma) \cap (C_H + C')$  for every compact subset  $C'$  of  $V$ .

Our proof is also based on the following observation (\*): if  $(x_n)_{n \in \mathbb{N}}$  is a sequence of points of  $V^+$  whose distance to  $C_H$  is larger than a given  $R > 0$ , and if  $\|x_{n+1} - x_n\| \leq R$  for all  $n \in \mathbb{N}$ , then all elements  $x_n$  belong to the same connected component of  $V^+ \setminus C_H$ .

We now give a proof of Theorem 1.2. Let  $C_1, \dots, C_s$  be the connected components of  $V^+ \setminus C_H$  and let  $\Gamma$  be a discrete subgroup of  $G$  acting properly discontinuously on  $G/H$ . The set  $\mu(\Gamma)$  is invariant under the opposition involution  $\iota$ .

Assume that  $\Gamma$  is not a torsion group and fix an element  $\gamma \in \Gamma$  of infinite order. Since  $\Gamma$  is discrete and since  $\mu$  is a proper map, the sequence  $(\|\mu(\gamma^n)\|)_{n \in \mathbb{Z}}$  tends to infinity as  $n$  tends to  $\pm\infty$ . Let  $F$  be the set of elements  $\gamma' \in \Gamma$  such that the distance of  $\mu(\gamma')$  to  $C_H$  is  $\leq \|\mu(\gamma)\|$ . From the discreteness of  $\Gamma$ , the properness of  $\mu$ , and the properness criterion, we deduce that  $F$  is finite. Moreover, by Lemma 2.3,

$$\|\mu(\gamma^{n+1}) - \mu(\gamma^n)\| \leq \|\mu(\gamma)\|$$

for all  $n \in \mathbb{Z}$ . By the observation (\*) above, there are integers  $1 \leq i, j \leq s$  such that  $\mu(\gamma^n) \in C_i$  (resp.  $\mu(\gamma^{-n}) \in C_j$ ) for almost all  $n \in \mathbb{N}$ . The opposition involution  $\iota$  interchanges  $C_i$  and  $C_j$ .

Note that for every  $\gamma' \in \Gamma$ , Lemma 2.3 implies

$$\|\mu(\gamma'\gamma^n) - \mu(\gamma^n)\| \leq \|\mu(\gamma')\|$$

for all  $n \in \mathbb{Z}$ . By the properness criterion,  $\mu(\gamma'\gamma^n) \in C_i$  and  $\mu(\gamma'\gamma^{-n}) \in C_j$  for almost all  $n \in \mathbb{N}$ .

First consider the case  $i = j$ . Let  $F'$  be the set of elements  $\gamma' \in \Gamma$  such that  $\mu(\gamma') \notin C_i$ . We claim that  $F'$  is finite. Indeed, let  $\gamma' \in F'$ . By Lemma 2.3,

$$\|\mu(\gamma'\gamma^{n+1}) - \mu(\gamma'\gamma^n)\| \leq \|\mu(\gamma)\|$$

for all  $n \in \mathbb{Z}$ . Moreover,  $\mu(\gamma') \notin C_i$ , and we have just seen that  $\mu(\gamma'\gamma^n) \in C_i$  for almost all  $n \in \mathbb{Z}$ . By the observation (\*) above, there is an integer  $n \in \mathbb{Z}$  such that  $\gamma'\gamma^n \in F$ . Therefore,  $F' \subset F\gamma^{\mathbb{Z}}$ . Since  $F$  is finite and since for every  $f \in F$  the element  $f\gamma^n$  belongs to  $C_i$  for almost all  $n \in \mathbb{Z}$ , the set  $F'$  is finite. This proves the claim.

Now consider the case  $i \neq j$ . We claim that the subgroup  $\gamma^{\mathbb{Z}}$  has finite index in  $\Gamma$ . Indeed, let  $\gamma' \in \Gamma$ . By Lemma 2.3,

$$\|\mu(\gamma'\gamma^{n+1}) - \mu(\gamma'\gamma^n)\| \leq \|\mu(\gamma)\|$$

for all  $n \in \mathbb{Z}$ . Moreover, we have seen that  $\mu(\gamma'\gamma^n) \in C_i$  and  $\mu(\gamma'\gamma^{-n}) \in C_j$  for almost all  $n \in \mathbb{N}$ . By the observation (\*) above, there is an integer  $n \in \mathbb{Z}$

such that  $\gamma'\gamma^n \in F$ . Therefore,  $\Gamma = F\gamma^{\mathbb{Z}}$ . Since  $F$  is finite,  $\gamma^{\mathbb{Z}}$  has finite index in  $\Gamma$ . This proves the claim and completes the proof of Theorem 1.2.

**3.2. Discrete torsion groups in characteristic zero.** In this subsection we show that when  $\mathbf{k}$  has characteristic zero, the assumption that  $\Gamma$  is not a torsion group may be removed from Theorem 1.2. When  $\Gamma$  is known to be finitely generated, this follows from Selberg's lemma ([Sel], Lem. 8). In general it is also true, based on the following lemma, which is probably well known.

**Lemma 3.1.** *Let  $\mathbf{k}$  be a local field of characteristic zero and  $\mathbf{G}$  a linear algebraic  $\mathbf{k}$ -group. If  $\mathbf{k}$  is a  $p$ -adic field, then every torsion subgroup of  $G$  is finite. If  $\mathbf{k} = \mathbb{R}$  or  $\mathbb{C}$ , then every discrete torsion subgroup of  $G$  is finite.*

*Proof.* Embed  $\mathbf{G}$  in  $\mathbf{GL}_n$  for some  $n \geq 1$ . Let  $\Gamma$  be a torsion subgroup of  $G$ . By a result of Schur ([CR], Th. 36.14),  $\Gamma$  contains a finite-index abelian subgroup  $\Gamma'$  whose elements are all semisimple. To show that  $\Gamma$  is finite, it is enough to prove the finiteness of  $\Gamma'$ .

Assume that  $\mathbf{k}$  is a  $p$ -adic field. The elements of  $\Gamma'$  are diagonalizable in a common basis over an algebraic closure of  $\mathbf{k}$ . For every  $\gamma \in \Gamma'$  the eigenvalues of  $\gamma$  are roots of unity; they generate a cyclotomic extension  $\mathbf{k}_\gamma$  of  $\mathbf{k}$ , and  $[\mathbf{k}_\gamma : \mathbf{k}] \leq n$  since the characteristic polynomial of  $\gamma$  has degree  $n$ . Now there are only finitely many cyclotomic extensions of  $\mathbf{k}$  of degree  $\leq n$  ([Neu], Chap. 2, Th. 7.12 & Prop. 7.13). Therefore the field generated by all extensions  $\mathbf{k}_\gamma$ ,  $\gamma \in \Gamma'$ , has finite degree over  $\mathbf{k}$ , hence contains only finitely many roots of unity ([Neu], Chap. 2, Prop. 5.7). This implies the finiteness of  $\Gamma'$ .

Assume that  $\mathbf{k} = \mathbb{R}$  or  $\mathbb{C}$  and that in addition  $\Gamma$  is discrete in  $G$ . The elements of  $\Gamma'$  are diagonalizable in a common basis over  $\mathbb{C}$ , and their eigenvalues are roots of unity. Since the group  $\mathbb{U}$  of complex numbers of modulus one is compact, every discrete subgroup of  $\mathbb{U}^n$  is finite. This implies the finiteness of  $\Gamma'$ .  $\square$

When  $\mathbf{k}$  has positive characteristic, there exist infinite discrete torsion subgroups in  $G$ . They all have a unipotent subgroup of finite index (this follows from [Tits], Prop. 2.8, for instance). Some of them do not satisfy the conclusions of Theorem 1.2: we will give an example of such a group in Section 5.2.

#### 4. AN APPLICATION TO $\mathrm{SL}_n(\mathbf{k})/\mathrm{SL}_{n-1}(\mathbf{k})$

In this section we discuss the case of  $G = \mathrm{SL}_n(\mathbf{k})$  and  $H = \mathrm{SL}_{n-1}(\mathbf{k})$ . We show how Theorem 1.2 implies Corollary 1.4.

Let  $\mathbf{G} = \mathbf{SL}_n$  for some integer  $n \geq 2$ . The group  $\mathbf{A}$  of diagonal matrices of determinant one is a maximal  $\mathbf{k}$ -split  $\mathbf{k}$ -torus of  $\mathbf{G}$ , which is its own centralizer, *i.e.*,  $\mathbf{Z} = \mathbf{A}$ . The corresponding roots are the linear forms  $\varepsilon_i - \varepsilon_j$ ,  $1 \leq i \neq j \leq n$ , where

$$\varepsilon_i(\mathrm{diag}(a_1, \dots, a_n)) = a_i.$$

A basis of the root system of  $\mathbf{A}$  in  $\mathbf{G}$  is given by the roots  $\varepsilon_i - \varepsilon_{i+1}$ , where  $1 \leq i \leq n-1$ . If  $\mathbf{k} = \mathbb{R}$  or  $\mathbb{C}$  (resp. if  $\mathbf{k}$  is nonarchimedean), the corresponding

positive Weyl chamber is

$$A^+ = \{\text{diag}(a_1, \dots, a_n) \in A, a_i \in ]0, +\infty[ \forall i \text{ and } a_1 \geq \dots \geq a_n\} \\ (\text{resp. } A^+ = \{\text{diag}(a_1, \dots, a_n) \in A, |a_1| \geq \dots \geq |a_n|\}).$$

Set  $K = \text{SO}(n)$  (resp.  $K = \text{SU}(n)$ , resp.  $K = \text{SL}_n(\mathcal{O})$ ) if  $\mathbf{k} = \mathbb{R}$  (resp. if  $\mathbf{k} = \mathbb{C}$ , resp. if  $\mathbf{k}$  is nonarchimedean). The Cartan decomposition  $G = KA^+K$  holds. If  $\mathbf{k} = \mathbb{R}$  (resp. if  $\mathbf{k} = \mathbb{C}$ ), it follows from the polar decomposition in  $\text{GL}_n(\mathbb{R})$  (resp. in  $\text{GL}_n(\mathbb{C})$ ) and from the reduction of symmetric (resp. of Hermitian) matrices; if  $\mathbf{k}$  is nonarchimedean, it follows from the structure theorem for finitely generated modules over a principal ideal domain. The real vector space

$$V = \{(x_1, \dots, x_n) \in \mathbb{R}^n, x_1 + \dots + x_n = 0\} \simeq \mathbb{R}^{n-1}$$

and its closed convex cone

$$V^+ = \{(x_1, \dots, x_n) \in V, x_1 \geq \dots \geq x_n\}$$

do not depend on  $\mathbf{k}$ . Let  $\mu : G \rightarrow V^+$  denote the Cartan projection relative to the Cartan decomposition  $G = KA^+K$ . If  $\mathbf{k} = \mathbb{R}$  or  $\mathbb{C}$ , then  $\mu(g) = (x_1, \dots, x_n)$ , where  $e^{2x_i}$  is the  $i$ -th eigenvalue of  ${}^tgg$ .

Let  $\mathbf{H} = \text{SL}_{n-1}$ , which we consider as a subgroup of  $\mathbf{G}$  by embedding  $(n-1) \times (n-1)$  matrices in the upper left corner of  $n \times n$  matrices. Then

$$C_H = \bigcup_{1 \leq i \leq n} \{(x_1, \dots, x_n) \in V^+, x_i = 0\}$$

and the connected components of  $V^+ \setminus C_H$  are the sets

$$C_i = \{(x_1, \dots, x_n) \in V^+, x_i > 0 > x_{i+1}\},$$

where  $1 \leq i \leq n-1$ . The opposition involution  $\iota : V^+ \rightarrow V^+$  is given by

$$\iota(x_1, \dots, x_n) = (-x_n, \dots, -x_1);$$

it maps  $C_i$  to  $C_{n-i}$  for all  $1 \leq i \leq n-1$ . Here is a reformulation of Theorem 1.2 in the present situation.

**Proposition 4.1.** *Let  $\Gamma$  be a discrete subgroup of  $\text{SL}_n(\mathbf{k})$  that acts properly discontinuously on  $\text{SL}_n(\mathbf{k})/\text{SL}_{n-1}(\mathbf{k})$  and that is not a torsion group. There exists an integer  $1 \leq i \leq n-1$  such that  $\mu(\gamma) \in C_i \cup C_{n-i}$  for almost all  $\gamma \in \Gamma$ . If  $\Gamma$  is not virtually cyclic, then  $C_i = C_{n-i}$ .*

Note that if  $n$  is odd, then  $C_i \neq C_{n-i}$  for all  $1 \leq i \leq n-1$ , which implies Corollary 1.4. Another consequence of Proposition 4.1 is the following.

**Corollary 4.2.** *Assume that  $n \geq 4$  is even. Let  $\Gamma$  be a discrete subgroup of  $\text{SL}_n(\mathbf{k})$  that acts properly discontinuously on  $\text{SL}_n(\mathbf{k})/\text{SL}_{n-1}(\mathbf{k})$  and that is not virtually cyclic. Every element  $\gamma \in \Gamma$  of infinite order has  $n/2$  eigenvalues  $t$  with  $|t| > 1$  and  $n/2$  eigenvalues  $t$  with  $|t| < 1$ , counting multiplicities.*

The eigenvalues of an element  $g \in \text{SL}_n(\mathbf{k})$  belong to some finite extension  $\mathbf{k}_g$  of  $\mathbf{k}$ ; in Corollary 4.2 we denote by  $|\cdot|$  the unique absolute value on  $\mathbf{k}_g$  extending the absolute value  $|\cdot|$  on  $\mathbf{k}$ . As above, replacing  $\mathbf{k}$  by  $\mathbf{k}_g$ , we obtain a Cartan decomposition  $\text{SL}_n(\mathbf{k}_g) = K_g A_g^+ K_g$  with  $K = K_g \cap \text{SL}_n(\mathbf{k})$  and  $A^+ = A_g^+ \cap \text{SL}_n(\mathbf{k})$ . The corresponding Cartan projection  $\mu_g : \text{SL}_n(\mathbf{k}_g) \rightarrow V^+$  extends  $\mu$ .

**Proof of Corollary 4.2.** We may assume that  $\Gamma$  is not a torsion group. Since the only connected component of  $V^+ \setminus C_H$  that is invariant under  $\iota$  is  $C_{n/2}$ , Proposition 4.1 implies that  $\mu(\gamma) \in C_{n/2}$  for almost all  $\gamma \in \Gamma$ . Fix an element  $\gamma \in \Gamma$  of infinite order. Since  $\Gamma$  is discrete and since  $\mu$  is a proper map,  $\|\mu(\gamma^m)\| \rightarrow +\infty$  as  $m \rightarrow +\infty$ . Therefore

$$\frac{1}{m} \mu(\gamma^m) \in C_{n/2}$$

for almost all  $m \geq 1$ . Let  $\lambda : \mathrm{SL}_n(\mathbf{k}) \rightarrow V^+$  be the *Lyapunov projection* of  $\mathrm{SL}_n(\mathbf{k})$ , mapping  $g \in \mathrm{SL}_n(\mathbf{k})$  to  $\mu_g(a_g)$ , where  $a_g \in \mathrm{SL}_n(\mathbf{k}_g)$  is any diagonal matrix whose entries are the eigenvalues of  $g$  counted with multiplicities. By [Be2], Cor. 2.5,

$$\lambda(\gamma) = \lim_{m \rightarrow +\infty} \frac{1}{m} \mu(\gamma^m).$$

Thus  $\lambda(\gamma)$  belongs to the closure of  $C_{n/2}$  in  $V^+$ . We claim that  $\lambda(\gamma) \notin C_H$ . Indeed, by [Be2], Lem. 4.6, there is a constant  $C_\gamma > 0$  such that for all  $m \geq 1$ ,

$$(4.1) \quad \|\lambda(\gamma^m) - \mu(\gamma^m)\| \leq C_\gamma.$$

If  $\lambda(\gamma) \in C_H$ , then  $\lambda(\gamma^m) = m\lambda(\gamma) \in C_H$  for all  $m \geq 1$ , so that (4.1) would contradict the properness criterion (see Section 3.1). This proves the claim. Therefore,  $\lambda(\gamma) \in C_{n/2}$ , which means that  $\gamma$  has  $n/2$  eigenvalues  $t$  with  $|t| > 1$  and  $n/2$  eigenvalues  $t$  with  $|t| < 1$ , counting multiplicities.  $\square$

## 5. AN APPLICATION TO $(G \times G)/\Delta_G$ IN THE RANK-ONE CASE

In this section we prove Theorem 1.3, we show that the hypothesis that  $\Gamma$  is not a torsion group is necessary in the case of a local field of positive characteristic, and we describe an application to three-dimensional quadrics over a local field.

**5.1. Proof of Theorem 1.3.** Assume that  $\mathrm{rank}_{\mathbf{k}}(\mathbf{G}) = 1$  and let  $\Delta_{\mathbf{G}}$  denote the diagonal of  $\mathbf{G} \times \mathbf{G}$ . Fix a Cartan projection  $\mu$  of  $G$  and let  $\mu_{\bullet} = \mu \times \mu$  be the corresponding Cartan projection of  $G \times G$ . We identify the cone  $V^+$  with  $\mathbb{R}^+ \times \mathbb{R}^+$ , and  $C_{\Delta_G}$  with the diagonal of  $\mathbb{R}^+ \times \mathbb{R}^+$ . There are two connected components in  $V^+ \setminus C_{\Delta_G}$ ; let  $C_+$  (resp.  $C_-$ ) denote the one above (resp. below) the diagonal. The opposition involution  $\iota$  is the identity.

We now give a proof of Theorem 1.3. Let  $\Gamma$  be a discrete subgroup of  $G \times G$  that acts properly discontinuously on  $(G \times G)/\Delta_G$  and that is not a torsion group. Since  $\iota$  is the identity, Theorem 1.2 implies that either  $\mu_{\bullet}(\gamma) \in C_+$  for almost all  $\gamma \in \Gamma$ , or  $\mu_{\bullet}(\gamma) \in C_-$  for almost all  $\gamma \in \Gamma$ . Up to switching the factors of  $G \times G$ , we may assume that  $\mu_{\bullet}(\gamma) \in C_-$  for almost all  $\gamma \in \Gamma$ .

Let  $\mathrm{pr}_1$  (resp.  $\mathrm{pr}_2$ ) denote the projection of  $\Gamma$  on the first (resp. second) factor of  $G \times G$ . The kernel  $F$  of  $\mathrm{pr}_1$  is finite. If  $\Gamma$  is residually finite, then  $\Gamma$  contains a normal finite-index subgroup  $\Gamma'$  such that  $\Gamma' \cap F$  is trivial. If  $\Gamma$  is torsion-free, then  $F$  is already trivial and we set  $\Gamma' = \Gamma$ . In both cases, if we set  $\Gamma_0 = \mathrm{pr}_1(\Gamma')$ , then  $\varphi = \mathrm{pr}_2 \circ \mathrm{pr}_1^{-1} : \Gamma_0 \rightarrow G$  is a group homomorphism and

$$\Gamma' = \{(g, \varphi(g)), g \in \Gamma_0\}.$$

Since  $\mu(\varphi(g)) < \mu(g)$  for almost all  $g \in \Gamma_0$ , the group  $\Gamma_0$  is discrete in  $G$ . Indeed, if it were not, then there would be a sequence  $(g_n)_{n \in \mathbb{N}}$  of pairwise

distinct points of  $\Gamma_0$  converging to 1. Since  $\Gamma$  is discrete in  $G \times G$  and since  $\mu$  is a proper map, the sequence  $(\mu(\varphi(g_n)))_{n \in \mathbb{N}}$  would tend to infinity. Therefore there would be infinitely many elements  $(g, \varphi(g)) \in \Gamma$  with  $\mu(\varphi(g)) \geq \mu(g)$ , contradicting the assumption that  $\mu_\bullet(\gamma) \in C_-$  for almost all  $\gamma \in \Gamma$ . This proves that  $\Gamma_0$  is discrete in  $G$ . Since  $\mu(\varphi(g)) < \mu(g)$  for almost all  $g \in \Gamma_0$ , the properness criterion (see Section 3.1) ensures that for all  $R > 0$ , almost all  $g \in \Gamma_0$  satisfy  $\mu(\varphi(g)) < \mu(g) - R$ .

Conversely, if there exist a discrete subgroup  $\Gamma_0$  of  $G$  and a group homomorphism  $\varphi : \Gamma_0 \rightarrow G$  satisfying the conditions of Theorem 1.3, then  $\Gamma$  acts properly discontinuously on  $(G \times G)/\Delta_G$  by the properness criterion.

**5.2. Infinite torsion groups in positive characteristic.** Take  $\mathbf{G} = \mathbf{SL}_2$  over  $\mathbf{k} = \mathbb{F}_q((t))$ , where  $\mathbb{F}_q$  is a finite field of characteristic  $p$ . We now give an example of an infinite discrete torsion subgroup of  $G \times G$  that acts properly discontinuously on  $(G \times G)/\Delta_G$  and nevertheless does not satisfy the conclusions of Theorems 1.2 and 1.3. The Cartan decomposition  $G = KA^+K$  holds, where  $K = \mathbf{SL}_2(\mathcal{O}) = \mathbf{SL}_2(\mathbb{F}_q[[t]])$  and where  $A^+$  is the set of diagonal matrices  $\text{diag}(a_1, a_2)$  of  $G$  with  $|a_1| \geq |a_2|$  (see Section 4). Let  $\mu$  be the corresponding Cartan projection. For every  $n \in \mathbb{N}$ , set

$$g_n = \begin{pmatrix} 1 & t^{-n} \\ 0 & 1 \end{pmatrix}.$$

Note that for  $1 \leq r \leq p-1$ ,

$$\mu(g_n^r) = 2n.$$

This can be seen by expanding  $g_n^r$  as follows:

$$g_n^r = \begin{pmatrix} 1 & rt^{-n} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r & 0 \\ t^n & r^{-1} \end{pmatrix} \begin{pmatrix} t^{-n} & 0 \\ 0 & t^n \end{pmatrix} \begin{pmatrix} r^{-1}t^n & 1 \\ -1 & 0 \end{pmatrix}.$$

Let  $\Gamma$  be the subgroup of  $G \times G$  generated by the elements  $(g_n, g_{2n})$  and the elements  $(g_{2n}, g_n)$ , where  $n \in \mathbb{N}$ . It is an infinite residually finite discrete subgroup of  $G$  and each of its nontrivial elements has order  $p$ . The group  $\Gamma$  acts properly discontinuously on  $(G \times G)/\Delta_G$  by the properness criterion (see Section 3.1). It is not virtually cyclic. However, the two connected components of  $V^+ \setminus C_{\Delta_G}$  both contain infinitely many points of the form  $\mu(\gamma)$ , where  $\gamma \in \Gamma$ .

### 5.3. An application to three-dimensional quadrics over a local field.

As was pointed out in the introduction, one of the motivations for our investigation of  $(G \times G)/\Delta_G$  in the rank-one case is its application to three-dimensional quadrics over a local field  $\mathbf{k}$ . We now discuss this point in more detail.

Let  $\mathbf{k}$  be a local field and  $Q$  be a quadratic form on  $\mathbf{k}^4$ . Consider the unit sphere

$$S(Q) = \{x \in \mathbf{k}^4, Q(x) = 1\}.$$

By Witt's theorem, it identifies with the homogeneous space  $\mathbf{SO}(Q)/\mathbf{H}$ , where  $\mathbf{SO}(Q)$  is the special orthogonal group of  $Q$  and  $\mathbf{H}$  is an algebraic  $\mathbf{k}$ -subgroup of  $\mathbf{SO}(Q)$  defined as the stabilizer of some point  $x \in S(Q)$ .

If  $Q$  is  $\mathbf{k}$ -anisotropic, then  $\mathrm{SO}(Q)$  is compact ([BoT], § 4.24); thus every discrete subgroup of  $\mathrm{SO}(Q)$  is finite and acts properly discontinuously on  $S(Q)$ .

Assume that  $Q$  has Witt index one, *i.e.*, that  $\mathrm{rank}_{\mathbf{k}}(\mathbf{SO}(Q)) = 1$ . If  $\mathbf{H}$  is  $\mathbf{k}$ -anisotropic, then  $H$  is compact, and every discrete subgroup of  $\mathrm{SO}(Q)$  acts properly discontinuously on  $S(Q)$ . On the other hand, if  $\mathrm{rank}_{\mathbf{k}}(\mathbf{H}) = 1$ , then every discrete subgroup of  $\mathrm{SO}(Q)$  acting properly discontinuously on  $S(Q)$  is finite: this is the Calabi-Markus phenomenon ([Ko1], Cor. 4.4). For instance, if  $\mathbf{k} = \mathbb{R}$ , then every quadratic form  $Q$  on  $\mathbf{k}^4$  of Witt index one is equivalent to  $x_1^2 - x_2^2 - x_3^2 - x_4^2$  or to  $x_1^2 + x_2^2 + x_3^2 - x_4^2$ . In the first case,  $\mathrm{SO}(Q)$  (resp.  $H$ ) is isomorphic to  $\mathrm{SO}(1, 3)$  (resp. to  $\mathrm{SO}(3)$ ) and every discrete subgroup of  $\mathrm{SO}(Q)$  acts properly discontinuously on  $S(Q)$ . In the second case,  $\mathrm{SO}(Q)$  (resp.  $H$ ) is isomorphic to  $\mathrm{SO}(3, 1)$  (resp. to  $\mathrm{SO}(2, 1)$ ) and every discrete subgroup of  $\mathrm{SO}(Q)$  acting properly discontinuously on  $S(Q)$  is finite.

Now assume that  $Q$  has Witt index two, *i.e.*, that  $\mathrm{rank}_{\mathbf{k}}(\mathbf{SO}(Q)) = 2$ . For instance, if  $\mathbf{k} = \mathbb{R}$ , then  $\mathrm{SO}(Q)$  (resp.  $H$ ) is isomorphic to  $\mathrm{SO}(2, 2)$  (resp. to  $\mathrm{SO}(1, 2)$ ). We may assume that  $Q$  is given by

$$Q(x_1, x_2, x_3, x_4) = x_1x_4 - x_2x_3$$

and that  $\mathbf{H}$  is the stabilizer of  $x = (1, 0, 0, 1) \in S(Q)$ . Note that there is a natural transitive action of the group  $\mathrm{SL}_2(\mathbf{k}) \times \mathrm{SL}_2(\mathbf{k})$  on  $S(Q)$ . Indeed,  $\mathrm{SL}_2(\mathbf{k}) \times \mathrm{SL}_2(\mathbf{k})$  acts on  $M_2(\mathbf{k})$  by the formula  $(g_1, g_2) \cdot u = g_1 u g_2^{-1}$  for all  $(g_1, g_2) \in \mathrm{SL}_2(\mathbf{k}) \times \mathrm{SL}_2(\mathbf{k})$  and all  $u \in M_2(\mathbf{k})$ ; identifying  $M_2(\mathbf{k})$  with  $\mathbf{k}^4$  gives a linear action of  $\mathrm{SL}_2(\mathbf{k}) \times \mathrm{SL}_2(\mathbf{k})$  on  $\mathbf{k}^4$  that preserves  $Q$  and is transitive on  $S(Q)$ . Since the stabilizer of  $x = (1, 0, 0, 1)$  in  $\mathrm{SL}_2(\mathbf{k}) \times \mathrm{SL}_2(\mathbf{k})$  is  $\Delta_{\mathrm{SL}_2(\mathbf{k})}$ , the quadric  $S(Q)$  identifies with the homogeneous space  $(\mathrm{SL}_2(\mathbf{k}) \times \mathrm{SL}_2(\mathbf{k})) / \Delta_{\mathrm{SL}_2(\mathbf{k})}$ . By Theorem 1.3, up to switching the factors of  $\mathrm{SL}_2(\mathbf{k}) \times \mathrm{SL}_2(\mathbf{k})$ , the torsion-free discrete subgroups  $\Gamma$  of  $\mathrm{SL}_2(\mathbf{k}) \times \mathrm{SL}_2(\mathbf{k})$  acting properly discontinuously on  $S(Q)$  are exactly the graphs of the form

$$\Gamma = \{(\gamma, \varphi(\gamma)), \gamma \in \Gamma_0\},$$

where  $\Gamma_0$  is a discrete subgroup of  $\mathrm{SL}_2(\mathbf{k})$  and  $\varphi : \Gamma_0 \rightarrow \mathrm{SL}_2(\mathbf{k})$  is a group homomorphism such that for all  $R > 0$ , almost all  $\gamma \in \Gamma_0$  satisfy  $\mu(\varphi(\gamma)) < \mu(\gamma) - R$ .

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